$$\mathbf{a} = \mathbf{a}_1 + \sum_{s=1}^{m} \chi_s(t)\boldsymbol{\psi}_s, \quad (\mathbf{a}_1 \cdot \boldsymbol{\psi}_s)_{H_1} = 0, \quad s = 1, \ldots, m$$

where $\boldsymbol{\psi}_s$ is the basis of the rigid displacement vectors ($m = 6$ in the case of an unfixed boundary).

Note 3.3. For each specific problem of the linear viscoelasticity problems posed, the domain of variation of the parameter β is bounded. For partial principle Y to be satisfied simultaneously for all such specific problems, it is necessary to require that all roots of the appropriate polynomials $P_s(p, \alpha, \beta)$ lie in the left half-plane of the complex variable p for all $\alpha \in [0, \alpha_s^\circ]$, $\beta \in R_+$.

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STATE OF STRESS IN A FLAT CIRCULAR RING WITH A CRACK

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The stress distribution in a circular isotropic ring with a crack on part of the concentric circle is investigated. A system of functional equations governing the coefficients of the complex Fourier series expansion of the stresses acting on the circle on which the crack is located is obtained. The solution of the mentioned system of equations is obtained by using a factorization method, which permitted reduction of the initial system of equations to two coupled infinite systems of algebraic equations. The possibility of using the method

of truncation to solve these systems is proved. The singularity originating in the neighborhood of ends of the crack in the formulas governing the stresses is isolated. Stress intensity coefficients for the effect of a uniform load on the external contour are presented.

1. Let us examine a flat circular ring $(a < r < b,\ |\theta| < \pi)$ with a crack $(r = c,\ |\theta| < \theta_0)$ loaded by normal and shear stresses on the outer and inner contours

$$\sigma_r(b,\theta) = \sum_{n=-\infty}^{\infty} p_n^+ e^{in\theta}, \quad \sigma_r(a,\theta) = \sum_{n=-\infty}^{\infty} p_n^- e^{in\theta} \tag{1.1}$$

$$\tau_{r\theta}(b,\theta) = \sum_{n=-\infty}^{\infty} q_n^+ e^{in\theta}, \quad \tau_{r\theta}(a,\theta) = \sum_{n=-\infty}^{\infty} q_n^- e^{in\theta}$$

in a polar coordinate system.

Not all the coefficients in these expansions are independent since there follows from the equilibrium condition for the load on each contour:

$$q_0^\pm = 0, \quad q_1^\pm = -ip_1^\pm, \quad q_{-1}^\pm = ip_{-1}^\pm$$

The determination of stresses in a ring is related to seeking the Airy function satisfying a biharmonic equation. This function can be represented as

$$F(r,\theta) = B_0 r^2 + C_0 \ln r + D_0 + (A_1 r^3 + C_1 r + D_1 r^{-1}) e^{i\theta} + \tag{1.2}$$
$$(A_{-1} r^3 + C_{-1} r + D_{-1} r^{-1}) e^{-i\theta} +$$
$$\sum_{|n|\geqslant 2} (A_n r^n + B_n r^{-n} + C_n r^{n+2} + D_n r^{-n+2}) e^{in\theta}$$

where $(B_0,\ C_0\ \ldots,\ A_n,\ \ldots,\ D_n$ are constants. The normal and shear stresses on the outer and inner contours must hence be equal to those given by (1.1) and zero on the crack.

Let us image the ring slit along the circle $r = c$ and consider two rings: an outer $(c < r < b,\ |\theta| < \pi)$ and an inner $(a < r < c,\ |\theta| < \pi)$. We represent the unknown stresses acting on the circle $r = c$ as

$$\sigma_r(c,\theta) = \sum_{n=-\infty}^{\infty} f_n e^{in\theta}, \quad \tau_{r\theta}(c,\theta) = \sum_{n=-\infty}^{\infty} \varphi_n e^{in\theta} \tag{1.3}$$

$$(\varphi_0 = 0,\ \varphi_1 = -if_1,\ \varphi_{-1} = if_{-1})$$

The Airy function for each of the two rings can be taken in the form (1.2) and the integration constants can be determined from the appropriate boundary conditions. These constants will be expressed in terms of the unknown coefficients f_n and φ_n of the expansion (1.3). To determine these latter, the conditions connecting the rings under consideration along their common boundary $r = c$ should be used, on which the following conditions must be satisfied:

$$\sigma_r(c,\ \theta) = \tau_{r\theta}(c,\ \theta) = 0 \quad (|\theta| < \theta_0) \tag{1.4}$$

$$u_r(c+0,\ \theta) = u_r(c-0,\ \theta)$$

$$u_\theta(c+0,\ \theta) = u_\theta(c-0,\ \theta) \quad (\pi > |\theta| > \theta_0)$$

where u_r, u_θ are the displacement vector components.

The connection conditions (1.4) permit obtaining the system

$$\sum_{n=-\infty}^{\infty} f_n e^{in\theta} = 0, \quad \sum_{n=-\infty}^{\infty} \varphi_n e^{in\theta} = 0 \quad (|\theta| < \theta_0) \tag{1.5}$$

$$\sum_{n=-\infty}^{\infty} \begin{Vmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{Vmatrix} \begin{Vmatrix} f_n \\ \varphi_n \end{Vmatrix} e^{in\theta} = \sum_{n=-\infty}^{\infty} \begin{Vmatrix} t_n \\ \tau_n \end{Vmatrix} e^{in\theta} - \tag{1.6}$$

$$2 \begin{Vmatrix} \sin\theta & \cos\theta \\ i\cos\theta & -i\sin\theta \end{Vmatrix} \begin{Vmatrix} \zeta_0 \\ \eta_0 \end{Vmatrix} \quad (\theta_0 < |\theta| < \pi)$$

Here

$$\alpha_0 = -2c(1-\nu_0)\frac{s^2-\varepsilon^2}{(s^2-1)(1-\varepsilon^2)}, \quad \beta_0 = \gamma_0 = \delta_0 = \beta_{\pm 1} = \delta_{\pm 1} = 0$$

$$\alpha_{\pm 1} = -c(1-2\nu_0)\frac{s^4-\varepsilon^4}{(s^4-1)(1-\varepsilon^4)}, \quad \gamma_{\pm 1} = \mp c(3-2\nu_0)\frac{s^4-\varepsilon^4}{(s^4-1)(1-\varepsilon^4)}$$

$$\begin{Vmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{Vmatrix} = \frac{c(1-\nu_0)s^2}{(n^2-1)D_n^+ D_n^-} \begin{Vmatrix} a_n & ib_n \\ c_n & id_n \end{Vmatrix}, \quad s = \frac{b}{c}, \quad \varepsilon = \frac{a}{c} \tag{1.7}$$

$$D_n^+(s, \varepsilon) = s^{2n} + s^{-2n} + 2(n^2-1) - n^2(s^2+s^{-2}), \ D_n^-(s, \varepsilon) = D_n^+(\varepsilon, s)\, s^2$$

$$a_n = 4n(s^{-2n}\varepsilon^{2n} - \varepsilon^{-2n}s^{2n}) + 4n(n^2-1)(s^{-2n} - s^{2n} + \varepsilon^{2n} - \varepsilon^{-2n}) + 2n^2s^2[(n-1)\varepsilon^{-2n} - (n+1)\varepsilon^{2n}] + 2n^2s^{-2}[(n+1)\varepsilon^{-2n} - (n-1)\varepsilon^{2n}] + 2n^2\varepsilon^{-2}[(n-1)s^{2n} - (n+1)s^{-2n}] + 2n^2\varepsilon^2[(n+1)s^{2n} - (n-1)s^{-2n}] + 4n^2(n^2-1)(s^{-2} - s^2 + \varepsilon^2 - \varepsilon^{-2}) + 4n^4(s^2\varepsilon^{-2} - s^{-2}\varepsilon^2)$$

$$b_n = c_n = 4(s^{2n}\varepsilon^{-2n} - s^{-2n}\varepsilon^{2n}) + 4n^3(\varepsilon^2s^{-2} - \varepsilon^{-2}s^2) + 2n^2s^2 \times [(n+1)\varepsilon^{2n} + (n-1)\varepsilon^{-2n}] - 2n^2\varepsilon^2[(n-1)s^{-2n} + (n+1)s^{2n}] + 4(n^2-1)[(n-1)s^{-2n} + (n+1)s^{2n}] - 4(n^2-1)[(n+1)\varepsilon^{2n} + (n-1)\varepsilon^{-2n}] + 2ns^{-2}[(n+2)(n-1)\varepsilon^{2n} + (n-2)(n+1)\varepsilon^{-2n}] + 8n(n^2-1)(\varepsilon^{-2} - s^{-2}) - 2n\varepsilon^{-2}[(n-2)(n+1)s^{-2n} + (n+2)(n-1)s^{2n}]$$

$$d_n = 4n(s^{-2n}\varepsilon^{2n} - s^{2n}\varepsilon^{-2n}) + 4n^2(n^2-2)(\varepsilon^2s^{-2} - \varepsilon^{-2}s^2) + 2n^2s^2[(n-1)\varepsilon^{-2n} - (n+1)\varepsilon^{2n}] + 2n^2\varepsilon^2[(n+1)s^{2n} - (n-1)s^{-2n}] - 2s^{-2}[(3n^2+n^3-4)\varepsilon^{2n} + (3n^2-n^3-4)\varepsilon^{-2n}] + 4n^2(n^2-1)(s^2-\varepsilon^2) + 2\varepsilon^{-2}[(3n^2-n^3-4)s^{-2n} + (3n^2+n^3-4)s^{2n}] + 4(n^2-1)(n^2-4)(\varepsilon^{-2} - s^{-2}) - 4(n^2-1)[(n+2)s^{2n} - (n-2)s^{-2n}] + 4(n^2-1)[(n+2)\varepsilon^{2n} - (n-2)\varepsilon^{-2n}]$$

$$\begin{aligned} -t_n &= U_n p_n^+ + V_n q_n^+ + G_n p_n^- + H_n q_n^- \\ -\tau_n &= K_n p_n^+ + L_n q_n^+ + M_n p_n^- + N_n q_n^- \end{aligned} \tag{1.8}$$

$$U_0 = 2c(1-\nu_0)\frac{s^2}{s^2-1}, \quad U_{\pm 1} = c(1-2\nu_0)\frac{s^3}{s^4-1}$$

$$G_0 = 2c(1-\nu_0)\frac{\varepsilon^2}{1-\varepsilon^2}, \quad G_{\pm 1} = c(1-2\nu_0)\frac{\varepsilon^3}{1-\varepsilon^4}$$

$$K_{\pm 1} = c(3-2\nu_0)\frac{s^3}{s^4-1}, \quad M_{\pm 1} = c(3-2\nu_0)\frac{\varepsilon^3}{1-\varepsilon^4}$$

$$V_0 = H_0 = K_0 = L_0 = M_0 = N_0 = V_{\pm 1} = H_{\pm 1} = L_{\pm 1} = N_{\pm 1} = 0$$

$$|n| \geqslant 2, \; U_n = n\rho_n^+ [(n+1)s^n + (n-1)s^{-n} - (n+1)s^{-n-2} - (n-1)s^{n-2}]$$

$$V_n = i\rho_n^+ [(n+1)(n-2)s^n - (n+2)(n-1)s^{-n} + n(n+1)\times s^{-n-2} - n(n-1)s^{n-2}]$$

$$G_n = n\rho_n^- [(n-1)\varepsilon^n + (n+1)\varepsilon^{-n} - (n-1)\varepsilon^{-n+2} - (n+1)\times \varepsilon^{n+2}]$$

$$H_n = i\rho_n^- [n(n-1)\varepsilon^n - n(n+1)\varepsilon^{-n} - (n+1)(n-2)\varepsilon^{n+2} + (n-1)(n+2)\varepsilon^{-n+2}]$$

$$K_n = \rho_n^+ [n(n-1)s^{-n} - n(n+1)s^n - (n+1)(n-2)s^{-n-2} + (n-1)(n+2)s^{n-2}]$$

$$L_n = i\rho_n^+ [(n-2)(n+1)s^{-n-2} + (n+2)(n-1)s^{n-2} - (n+2)(n-1)s^{-n} - (n-2)(n+1)s^n]$$

$$M_n = \rho_n^- [(n+1)(n-2)\varepsilon^{-n} - (n-1)(n+2)\varepsilon^n - n(n-1)\varepsilon^{-n+2} + n(n+1)\varepsilon^{n+2}]$$

$$N_n = i\rho_n^- [(n-2)(n+1)\varepsilon^{n+2} + (n+2)(n-1)\varepsilon^{-n+2} - (n+2)(n-1)\varepsilon^n - (n-2)(n+1)\varepsilon^{-n}]$$

$$\rho_n^{\pm} = 2c(1-\nu_0)\frac{s^2}{(n^2-1)D_n^{\pm}}$$

($\nu_0 = \nu / (1+\nu)$ for the plane stress, $\nu_0 = \nu$ for plane strain, ν is the Poisson ratio, and ζ_0, η_0 are constants governing the ring displacement as a solid body).

Let us require the satisfaction of estimate

$$\sigma_r, \; \tau_{r\theta} = O(r^{-1/2}), \qquad r \to 0 \tag{1.9}$$

at the ends of the crack. Then, in a standard manner, it is easy to show that the problem has a unique solution. The series coefficients for the stresses have

$$f_n, \; \varphi_n = O(|n|^{-1/2}), \qquad |n| \to \infty \tag{1.10}$$

Consequently, the series (1.3) converge nonuniformly. Because of this nature of the convergence, subsequent transformations are formal in nature. However, the series governing the stresses on the circle $r = c$ are converted in that form for which the singularity (1.9) is extracted explicitly, and the remainder is represented by uniformly convergent series. After this, compliance with all the boundary conditions can be verified rigorously.

2. Let us represent the matrix (1.7) as the sum

$$\begin{Vmatrix}\alpha_n & \beta_n\\ \gamma_n & \delta_n\end{Vmatrix} = \frac{\lambda}{n^2-1}\begin{Vmatrix}-|n| & i\operatorname{sgn} n\\ \operatorname{sgn} n & -i|n|\end{Vmatrix} + \frac{\lambda}{n^2-1}\begin{Vmatrix}a_n^\circ & b_n^\circ\\ c_n^\circ & d_n^\circ\end{Vmatrix} \tag{2.1}$$
$$\lambda = 4c(1-\nu_\theta)$$

The first matrix in the right side in (2.1) determines the coefficients $\alpha_n, \ldots, \delta_n$ in the case of an infinite domain (this is the principal part of the matrix), and the elements of the second matrix tend exponentially to zero as $|n| \to \infty$.

We substitute the matrix (1.7), written as the sum (2.1), into the functional equation (1.6) and we transfer terms containing elements of the second matrix in this sum to the right side. (If the elements of the second matrix are assumed zero and the system (1.5) and (1.6) is solved, then we obtain the solution to the problem for an infinite plate with a crack under an arbitrary load in a form different from [1]).

We introduce new unknowns by means of the formula

$$\frac{1}{n^2-1}\begin{Vmatrix}-|n| & i\operatorname{sgn} n\\ \operatorname{sgn} n & -i|n|\end{Vmatrix}\begin{Vmatrix}f_n\\ \varphi_n\end{Vmatrix} = \begin{Vmatrix}f_n^\circ\\ \varphi_n^\circ\end{Vmatrix}, \quad |n|\geqslant 2 \tag{2.2}$$

Hence, as follows from (1.10)

$$f_n^\circ,\ \varphi_n^\circ = O\left(|n|^{-3/2}\right), \quad |n|\to\infty \tag{2.3}$$

Taking account of (2.1), (2.2), the functional equations (1.5), (1.6) are converted into

$$\sum_{|n|\geqslant 2}\left(-|n|f_n^\circ - \operatorname{sgn} n\varphi_n^\circ\right)e^{in\theta} = -f_0 - f_1e^{i\theta} - f_{-1}e^{-i\theta} \tag{2.4}$$

$$\sum_{|n|\geqslant 2}\left(\operatorname{sgn} nf_n^\circ + |n|\varphi_n^\circ\right)e^{in\theta} = f_1e^{i\theta} - f_{-1}e^{-i\theta} \quad (|\theta| < \theta_0)$$

$$\lambda\sum_{|n|\geqslant 2} f_n^\circ e^{in\theta} = -\alpha_0f_0 - \alpha_1f_1e^{i\theta} - \alpha_{-1}f_{-1}e^{-i\theta} + \tag{2.5}$$
$$\sum_{n=-\infty}^{\infty}\psi_ne^{in\theta} - 2(\zeta_0\sin\theta + \eta_0\cos\theta)$$
$$\lambda\sum_{|n|\geqslant 2}\varphi_n^\circ e^{in\theta} = -\gamma_1f_1e^{i\theta} - \gamma_{-1}f_{-1}e^{-i\theta} +$$
$$\sum_{n=-\infty}^{\infty}\omega_ne^{in\theta} + 2i(\zeta_0\cos\theta - \eta_0\sin\theta) \quad (\theta_0 < |\theta| < \pi)$$

Here

$$\psi_n = t_n, \quad \omega_n = \tau_n, \quad |n| < 2 \tag{2.6}$$
$$\begin{Vmatrix}\psi_n\\ \omega_n\end{Vmatrix} = \begin{Vmatrix}t_n\\ \tau_n\end{Vmatrix} - \frac{\lambda}{n^2-1}\begin{Vmatrix}a_n^\circ & b_n^\circ\\ c_n^\circ & d_n^\circ\end{Vmatrix}\begin{Vmatrix}-|n|f_n^\circ - \operatorname{sgn} n\varphi_n^\circ\\ i\operatorname{sgn} nf_n^\circ + i|n|\varphi_n^\circ\end{Vmatrix}, \quad |n|\geqslant 2$$

Let us separate the external load into symmetric and antisymmetric parts in the angle θ. We examine the case of the effect of the first of the loads mentioned when

$$p_n^\pm = p_{-n}^\pm, \quad q_n^\pm = -q_{-n}^\pm, \quad f_n = f_{-n}, \quad \varphi_n = -\varphi_{-n}, \quad f_n^\circ = f_{-n}^\circ, \tag{2.7}$$
$$\varphi_n^\circ = -\varphi_{-n}^\circ$$

The second case for which the signs in the right sides of (2. 7) are reversed, is considered analogously.

Taking account of (2. 7), the functional equations (2. 4), (2. 5) are converted into

$$\sum_{n\geqslant 2} (nf_n^\circ + \varphi_n^\circ)\cos n\theta = \frac{1}{2} f_0 + f_1 \cos\theta \tag{2.8}$$

$$\sum_{n\geqslant 2} (f_n^\circ + n\varphi_n^\circ)\sin n\theta = f_1 \sin\theta \quad (|\theta| < \theta_0)$$

$$\lambda \sum_{n\geqslant 2} f_n^\circ \cos n\theta = -\frac{1}{2}\alpha_0 f_0 - \bar{\alpha}_1 f_1 \cos\theta + \frac{1}{2}\psi_0 + \tag{2.9}$$

$$\sum_{n\geqslant 1} \psi_n \cos n\theta - \eta_0 \cos\theta$$

$$\lambda \sum_{n\geqslant 2} \varphi_n^\circ \sin n\theta = -\gamma_1 f_1 \sin\theta + \sum_{n\geqslant 1} \omega_n \sin n\theta + \eta_0 \sin\theta$$

$$(\theta_0 < |\theta| < \pi)$$

The unknown coefficients in (2. 8) are not separated. To separate them, we consider the functions $P(\theta)$ and $Q(\theta)$

$$\sum_{n\geqslant 2} f_n^\circ \sin n\theta = P(\theta), \quad \sum_{n\geqslant 2} \varphi_n^\circ \cos n\theta = Q(\theta) \tag{2.10}$$

Hence

$$\sum_{n\geqslant 2} nf_n^\circ \cos n\theta = \frac{dP(\theta)}{d\theta}, \quad \sum_{n\geqslant 2} -n\varphi_n^\circ \sin n\theta = \frac{dQ(\theta)}{d\theta} \tag{2.11}$$

Substituting (2. 10) and (2. 11) into (2. 8), integrating and taking into account that $P(0) = 0$, we find

$$P(\theta) = -\mu \sin\theta, \quad Q(\theta) = \mu\cos\theta + {}^1/_2 f_0 + f_1 \cos\theta \tag{2.12}$$

where μ is a constant of integration. Therefore, the system of functional equations (2. 8) is written as

$$\sum_{n\geqslant 2} \varphi_n^\circ \cos n\theta = \mu\cos\theta + \frac{1}{2} f_0 + f_1 \cos\theta \tag{2.13}$$

$$\sum_{n\geqslant 2} f_n^\circ \sin n\theta = -\mu \sin\theta \quad (|\theta| < \theta_0)$$

We note that (2. 13) can be obtained directly from the boundary conditions on the crack if they were written as the vector of the forces acting on an arbitrary part of its length being equal to zero.

The system (2. 9), (2. 13) is similar to the system examined in [2] and admits an exact solution by the factorization method when the right side is known.

3. Let us introduce two analytic functions of the complex variable z

$$F_+(z) = \sum_{n\geqslant 2} f_n^\circ z^n, \quad \Phi_+(z) = \sum_{n\geqslant 2} \varphi_n^\circ z^n \tag{3.1}$$

The regularity of these functions in the circle $|z| < 1$ and the continuity up to the boundary follows from the estimate (2. 3). If we set

$$F_+(z^{-1}) = F_-(z), \qquad \Phi_+(z^{-1}) = \Phi_-(z) \tag{3.2}$$

(the functions $F_-(z)$ and $\Phi_-(z)$ are regular outside the unit circle and continuous up to its boundary), then the functional equations (2.9), (2.13) can be written as

$$F_+(e^{i\theta}) - F_-(e^{i\theta}) = -2i\mu\sin\theta \tag{3.3}$$

$$\Phi_+(e^{i\theta}) + \Phi_-(e^{i\theta}) = 2(\mu + f_1)\cos\theta + f_0 \quad (|\theta| < \theta_0)$$

$$\lambda[F_+(e^{i\theta}) + F_-(e^{i\theta})] = \psi_0 - \alpha_0 f_0 + 2\sum_{n\geqslant 1}\psi_n\cos n\theta - \tag{3.4}$$

$$2(\alpha_1 f_1 + \eta_0)\cos\theta$$

$$\lambda[\Phi_+(e^{i\theta}) - \Phi_-(e^{i\theta})] = 2i\Big[\sum_{n\geqslant 1}\omega_n\sin n\theta - (\gamma_1 f_1 - \eta_0)\sin\theta\Big]$$

$$(\theta_0 < |\theta| < \pi)$$

It follows from (3.3) and (3.4) that the solution of the problem posed reduces to the solution of Riemann-Hilbert problems of analytic function theory: find functions $F_+(z)$ and $\Phi_+(z)$ which are regular within the unit circle, and functions $F_-(z)$ and $\Phi_-(z)$ which are regular outside the unit circle by means of the relationship between their limit values of the boundary.

For convenience in the subsequent solution, we introduce new unknown functions $X_\pm(z)$ and $Y_\pm(z)$ by means of the following formulas:

$$X_+(z) = \lambda\Phi_+(z) + (\gamma_1 f_1 - \eta_0)z - \sum_{n\geqslant 1}\omega_n z^n \tag{3.5}$$

$$Y_+(z) = \lambda F_+(z) + (\alpha_1 f_1 + \eta_0)z - \frac{1}{2}(\psi_0 - \alpha_0 f_0) - \sum_{n\geqslant 1}\psi_n z^n$$

$$X_-(z) = X_+(z^{-1}), \quad Y_-(z) = -Y_+(z^{-1})$$

where the functions $X_+(z)$ and $Y_+(z)$ are regular in the domain $|z| < 1$ and the functions $X_-(z)$ and $Y_-(z)$ in $|z| > 1$).

Then (3.3) and (3.4) are converted into the form

$$X_+(e^{i\theta}) + X_-(e^{i\theta}) = \lambda[f_0 + 2(\mu + f_1)\cos\theta] + \tag{3.6}$$

$$2(\gamma_1 f_1 - \eta_0)\cos\theta - 2\sum_{n\geqslant 1}\omega_n\cos n\theta$$

$$Y_+(e^{i\theta}) + Y_-(e^{i\theta}) = 2i\Big\{[\alpha_1 f_1 + \eta_0 - \lambda\mu]\sin\theta - \sum_{n\geqslant 1}\psi_n\sin n\theta\Big\}$$

$$(|\theta| < \theta_0)$$

$$X_+(e^{i\theta}) - X_-(e^{i\theta}) = 0, \quad Y_+(e^{i\theta}) - Y_-(e^{i\theta}) = 0 \quad (|\theta_0 < |\theta| < \pi) \tag{3.7}$$

The functions $X_\pm(z)$ and $Y_\pm(z)$ are hence bounded at the ends of the arc

$$\sigma = \{|z| = 1, \ |\arg z| < \theta_0\}$$

As follows from (3.7), the functions $X_+(z)$ and $X_-(z)$ form a single analytic function which is regular in the z-plane with a slit along the arc σ. The same is valid for $Y_+(z)$ and $Y_-(z)$. Hence their limit values are connected by means of (3.6), which are solved in a known manner [3], consequently

$$X(z) = \frac{R(z)}{2\pi i}\int_\sigma \frac{d\tau}{\tau - z}\frac{1}{R(\tau)}\Big\{\lambda f_0 + [\lambda\mu + \gamma_1 f_1 + \lambda f_1 - \eta_0](\tau + \tau^{-1}) - \tag{3.8}$$

$$\left.\sum_{n\geqslant 1} \omega_n (\tau^n + \tau^{-n})\right\} = \begin{cases} X_+(z), & |z| < 1 \\ X_-(z), & |z| > 1 \end{cases}$$

$$Y(z) = \frac{R(z)}{2\pi i} \int_\sigma \frac{d\tau}{\tau - z} \frac{1}{R(\tau)} \Big\{[\alpha_1 f_1 + \eta_0 - \lambda\mu](\tau - \tau^{-1}) -$$

$$\left.\sum_{n\geqslant 1} \psi_n (\tau^n - \tau^{-n})\right\} = \begin{cases} Y_+(z), & |z| < 1 \\ Y_-(z), & |z| > 1 \end{cases}$$

$$R(z) = \sqrt{(z - e^{i\theta_0})(z - e^{-i\theta_0})}$$

(the integration is performed over the inner edge of σ). The function $R(z)$ is regular in the complex z-plane with a slit along σ where $R(0) = 1$.

The representations (3.8) yield the exact solution of the boundary value problem described by the system (2.9), (2.13).

4. The unknown coefficients f_0, f_1, η_0, μ, f_n°, φ_n° enter into the right side of (3.6). Let us proceed as follows to determine them. We expand the functions $X_+(z)$ and $Y_+(z)$, defined by (3.8) in Taylor series, and taking account of (3.5) we substitute these expansions into Eqs. (3.1). Then equating coefficients of identical powers of z, we obtain a system of equations determining the mentioned unknowns.

It is easy to note that to expand the functions (3.8) in Taylor series it is sufficient to obtain the expansion of the functions

$$S_m(z) = \frac{R(z)}{2\pi i} \int_\sigma \frac{d\tau}{\tau - z} \frac{\tau^m}{R(\tau)} \qquad (m = 0, \pm 1, \ldots) \tag{4.1}$$

into the mentioned series.

We note that the function $R(z)$ can be represented as a series [2]

$$R(z) = \sum_{p\geqslant 0} \rho_p(u) z^p, \qquad u = \cos\theta_0$$

$$\rho_n(u) = P_n(u) - 2uP_{n-1}(u) + P_{n-2}(u), \qquad \rho_0 = 1, \ \rho_1 = -u$$

where $P_n(u)$ is the Legendre polynomial.

Taking into account that for $|z| < 1$

$$\frac{1}{\tau - z} = \sum_{k\geqslant 0} z^k \tau^{-k-1}$$

we convert (4.1) into the form

$$S_m(z) = \sum_{n\geqslant 0} t_{mn} z^n, \qquad t_{mn} = \sum_{k=0}^{n} \rho_{n-k}(u) \frac{1}{2\pi i} \int_\sigma \frac{\tau^{m-k-1}}{R(\tau)} d\tau$$

Using the Mehler-Dirichlet formula for Legendre polynomials [4]

$$P_s(u) = \frac{1}{\pi} \int_{-\theta_0}^{\theta_0} \frac{d\varphi}{R(e^{i\varphi})} e^{-is\varphi} \tag{4.2}$$

we find

$$t_{mn} = \frac{1}{2} \sum_{k=0}^{n} \rho_{n-k}(u) P_{k-m}(u) \qquad \begin{pmatrix} m = 0, \pm 1, \ldots \\ n = 0, 1, \ldots \end{pmatrix} \tag{4.3}$$

As is shown in [2], for $m \neq n$ the last sum can be converted into

$$t_{mn}(u) = \frac{1}{2}\frac{m}{m-n}[P_{m-1}(u)P_n(u) - P_m(u)P_{n-1}(u)] \tag{4.4}$$

where $(n \neq 0)$.

Performing the above-mentioned substitution into (3. 1), we obtain the following system of equations (summation is over $k \geqslant 1$):

$$\lambda f_n^\circ = \psi_n + (\alpha_1 f_1 - \lambda\mu + \eta_0)(t_{1n} - t_{-1n}) - \Sigma\psi_k(t_{kn} - t_{-kn}) \tag{4.5}$$

$$\lambda\varphi_n^\circ = \omega_n + (\lambda\mu + \lambda f_1 + \gamma_1 f_1 - \eta_0)(t_{1n} + t_{-1n}) - \Sigma\omega_k(t_{kn} + t_{-kn}) \quad n \geqslant 2$$

$$A_0 Z = D \tag{4.6}$$

$$A_0 = \begin{Vmatrix} {}^1/{}_2\alpha_0 & -\alpha_1(t_{10} - t_{-10}) & -(t_{10} - t_{-10}) & \lambda(t_{10} - t_{-10}) \\ 0 & \alpha_1(1 - t_{11} + t_{-11}) & 1 - t_{11} + t_{-11} & \lambda(t_{11} - t_{-11}) \\ \lambda t_{00} & (\lambda + \gamma_1)(t_{10} + t_{-10}) & -(t_{10} + t_{-10}) & \lambda(t_{10} + t_{-10}) \\ 0 & \gamma_1 - (\lambda + \gamma_1)(t_{11} + t_{-11}) & -1 + t_{11} + t_{-11} & -\lambda(t_{11} + t_{-11}) \end{Vmatrix}$$

$$Z = \begin{Vmatrix} f_0 \\ f_1 \\ \eta_0 \\ \mu \end{Vmatrix}, \quad D = \begin{Vmatrix} \frac{1}{2}\psi_0 - \sum\psi_k(t_{k0} - t_{-k0}) \\ \psi_1 - \sum\psi_k(t_{k1} - t_{-k1}) \\ \sum\omega_k(t_{k0} + t_{-k0}) \\ \omega_1 - \sum\omega_k(t_{k1} + t_{-k1}) \end{Vmatrix}$$

The coefficients ψ_n, ω_n are expressed linearly in terms of the external load and the coefficients f_n°, φ_n° (see (2. 6)), hence the relationships (4. 5) are two coupled infinite systems of linear algebraic equations for f_n° and φ_n° (h_n^1 and h_n^2 are expressed in terms of the external load and the coefficients f_1, η_0, μ, and δ_{nm} is the Kronecker delta)

$$\begin{Vmatrix} f_n^\circ \\ \varphi_n^\circ \end{Vmatrix} = \sum_{m\geqslant 2}\begin{Vmatrix} R_{nm}^{11} & R_{nm}^{12} \\ R_{nm}^{21} & R_{nm}^{22} \end{Vmatrix}\begin{Vmatrix} f_m^\circ \\ \varphi_m^\circ \end{Vmatrix} + \begin{Vmatrix} h_n^1 \\ h_n^2 \end{Vmatrix}, \quad n \geqslant 2 \tag{4.7}$$

$$R_{nm}^{11} = K_{nm}^-(-ma_m^\circ + ib_m^\circ) - \delta_{nm}\frac{1}{n^2-1}(-na_n^\circ + ib_n^\circ)$$

$$R_{nm}^{12} = K_{nm}^-(-a_m^\circ + imb_m^\circ) - \delta_{nm}\frac{1}{n^2-1}(-a_n^\circ + inb_n^\circ)$$

$$R_{nm}^{21} = K_{nm}^+(-mc_m^\circ + id_m^\circ) - \delta_{nm}\frac{1}{n^2-1}(-nc_n^\circ + id_n^\circ)$$

$$R_{nm}^{22} = K_{nm}^+(-c_m^\circ + imd_m^\circ) - \delta_{nm}\frac{1}{n^2-1}(-c_n^\circ + ind_n^\circ)$$

$$K_{nm}^\pm = (t_{mn} \pm t_{-mn})\frac{1}{m^2-1}$$

5. We investigate the properties of the operator generated by the matrix $\|R_{nm}^{\alpha\beta}\|$ ($n, m \geqslant 2$). Let us first estimate the quantity t_{mn}. We have

$$t_{mn} = m[P_{n-1}\gamma_{mn} - P_n\gamma_{m-1,n-1}], \quad \gamma_{mn} = \frac{P_m - P_n}{m-n} \tag{5.1}$$

Using the representation (4. 2) for the Legendre polynomials, we find

$$\Upsilon_{mn} = -\frac{i}{\pi}\int_{-\theta_0}^{\theta_0} \varphi \frac{d\varphi}{R(e^{i\varphi})}\left(\frac{m-n}{2}\varphi\right)^{-1} \sin\frac{m-n}{2}\varphi \exp\left(-i\frac{m+n}{2}\varphi\right)$$

from which

$$|\Upsilon_{mn}| \leqslant \frac{1}{\pi}\int_{-\theta_0}^{\theta_0} \frac{|\varphi|\,d\varphi}{\sqrt{2(\cos\varphi - \cos\theta_0)}} = C \tag{5.2}$$

Here and below C denotes different constants whose exact values are not important.

Taking into account the estimate for Legendre polynomials [4] $|P_n(u)| \leqslant C / \sqrt{n}$ and also (5.2), we find

$$|t_{mn}| \leqslant Cm / \sqrt{n}$$

The relationships obtained when using the formulas for $a_n^\circ, \ldots, d_n^\circ$ (see (1.7) and (2.1)) result in the estimate

$$|R_{nm}^{\alpha\beta}| \leqslant Cm^3\varkappa^m / \sqrt{n} + Cn^2\sqrt{n}\varkappa^n\delta_{nm}, \quad \varkappa = \max(\varepsilon, s^{-1}) \tag{5.3}$$

It can also be verified that the free members of the system of equations are

$$h_n^1, \quad h_n^2 \to 0, \quad n \to \infty \tag{5.4}$$

It follows from the above that in a space of bounded sequences $\mathbf{x} = (x_2, \ldots, x_n, \ldots)$ with the norm $\|\mathbf{x}\| = \sup_n |x_n|$ the system of equations (4.7) can be considered as an equation of the second kind with a bounded operator R (we recall that $\varkappa < 1$)

$$\mathbf{x} + R\mathbf{x} = \mathbf{h}, \quad \|R\| \leqslant C \sup_{n\geqslant 2}\left[\frac{1}{\sqrt{n}}\sum_{m\geqslant 2} m^3\varkappa^m + n^2\sqrt{n}\varkappa^n\right] \tag{5.5}$$

It is clear that $\|R\| < 1$ for sufficiently small $\varkappa$ and then (5.5) is uniquely solvable, by, say, iterations. However, even for any $\varkappa < 1$ the estimates (5.3) and (5.4) permit the infinite system (5.5) to be referred to the class of quasi-regular systems [5] whose solution reduces to a successive solution of an infinite system with a small operator and a finite $N \times N$ system. Here N is determined from the condition

$$C\frac{1}{\sqrt{n}}\sum_{m\geqslant 2} m^3\varkappa^m + Cn^2\sqrt{n}\varkappa^n < 1 \quad \text{for } n > N$$

The unique solvability of this finite system follows directly from the uniqueness of the solution.

We substitute the coefficients f_n^0, φ_n^0 found from the system (4.5), into the system (4.6). Since these coefficients are expressed in terms of the constants f_1, μ, η_0, then we obtain a system of 4×4 equations determining all the coefficients needed.

6. Let us turn to extracting the singularities (1.9) in the series (1.3) for the stresses on the circle $r = c$.

We substitute f_n, φ_n from (2.2) into these series and use (4.7). To extract the singularity it is hence sufficient to limit to terms of the order of $O(n^{-1/2})$ as $n \to \infty$.

Let $\pi > \theta > \theta_0$. We have

$$\sigma_r(c, \theta) = -2\sum_{n\geqslant 2} n\cos n\theta \sum_{m\geqslant 2}(t_{mn} - t_{-mn}) \times \tag{6.1}$$

$$(Q_m^{11}f_m^\circ + Q_m^{12}\varphi_m^\circ) - 2\sum_{n\geqslant 2} n\cos n\theta \frac{A}{\lambda}(t_{1n} - t_{-1n}) +$$

$$2\sum_{n\geqslant 2} n\cos n\theta \sum_{m\geqslant 1}(t_{mn} - t_{-mn})\, t_m + \ldots$$

$$\tau_{r\theta}(c,\theta) = -2\sum_{n\geqslant 2} n\sin n\theta \sum_{m\geqslant 2}(t_{mn}+t_{-mn})(Q_m{}^{21}f_m{}^\circ + Q_m{}^{22}\varphi_m{}^\circ) -$$
$$2\frac{B}{\lambda}\sum_{n\geqslant 2} n\sin n\theta\,(t_{1n}+t_{-1n}) + 2\sum_{n\geqslant 2} n\sin n\theta \sum_{m\geqslant 1}(t_{mn}+t_{-mn})\,\tau_m + \ldots$$

Here

$$\left\|\begin{matrix} Q_m{}^{11} \\ Q_m{}^{12}\end{matrix}\right\| = \left\|\begin{matrix} -m & i \\ -1 & im\end{matrix}\right\| \left\|\begin{matrix} a_m{}^\circ \\ b_m{}^\circ\end{matrix}\right\|, \quad \left\|\begin{matrix} Q_m{}^{21} \\ Q_m{}^{22}\end{matrix}\right\| = \left\|\begin{matrix} -m & i \\ -1 & im\end{matrix}\right\| \left\|\begin{matrix} c_m{}^\circ \\ d_m{}^\circ\end{matrix}\right\|$$

$$A = \alpha_1 f_1 + \eta_0 - \lambda\mu, \quad B = \lambda\mu + (\gamma_1+\lambda) f_1 - \eta_0$$

Here and below, a series of dots will indicate uniformly convergent series which yield no contribution to the stress singularity.

Table 1

θ_0, deg	10		30		60		90	
ε	G_1	G_2	G_1	G_2	G_1	G_2	G_1	G_2
0.1	0.418	–0.037	0.667	–0.178	0.661	–0.394	0.454	–0.489
0.2	0.438	–0.038	0.755	–0.198	0.698	–0.459	0.403	–0.537
0.3	0.479	–0.041	0.935	–0.243	0.724	–0.575	0.318	–0.613
0.4	0.565	–0.046	1.263	–0.396	0.672	–0.753	0.171	–0.712

Let us introduce a series expansion for the generating function of the Legendre polynomials. Setting $z = e^{i\theta}$ and considering $\pi > \theta > \theta_0$, we find

$$\frac{ie^{-i\theta/2}}{\sqrt{2(u-\cos\theta)}} = \sum_{k\geqslant 0} P_k(u)\,e^{ik\theta}, \quad u = \cos\theta_0 \tag{6.2}$$

Taking account of (4.3) for t_{mn} and the expansion (6.2), we obtain

$$\sum_{n\geqslant 2} n\cos n\theta t_{mn} = -\frac{1}{\sqrt{2(u-\cos\theta)}}\frac{m}{2}\sin\frac{\theta}{2}\,[P_m(u)+P_{m-1}(u)] + \ldots$$
$$\sum_{n\geqslant 2} n\sin n\theta t_{mn} = \frac{1}{\sqrt{2(u-\cos\theta)}}\frac{m}{2}\cos\frac{\theta}{2}\,[P_m(u)-P_{m-1}(u)] + \ldots$$

Then for $\pi > \theta > \theta_0$ there follows from the representation (6.1)

$$\sigma_r(c,\theta) + i\tau_{r\theta}(c,\theta) = \frac{1}{\sqrt{2(\cos\theta_0-\cos\theta)}}\left[T_1\sin\frac{\theta}{2} + iT_2\cos\frac{\theta}{2}\right] + \ldots \tag{6.3}$$

$$T_1 = 2\sum_{m\geqslant 2}(Q_m{}^{11}f_m{}^\circ + Q_m{}^{12}\varphi_m{}^\circ)\,m\,(P_m+P_{m-1}) + 2\frac{A}{\lambda}(1+\cos\theta_0) -$$
$$2\sum_{m\geqslant 1} t_m m\,(P_m+P_{m-1})$$

$$T_2 = -2\sum_{m\geqslant 2}(Q_m{}^{21}f_m{}^\circ + Q_m{}^{22}\varphi_m{}^\circ)\,m\,(P_m-P_{m-1}) +$$
$$2\frac{B}{\lambda}(1-\cos\theta_0) + 2\sum_{m\geqslant 1}\tau_m m\,(P_m-P_{m-1})$$

Formulas (6.3) permit evaluation of the stress intensity coefficients [6]. In the case when a uniform pressure p_0^+ acts on the external contour and $\varepsilon = 1/s$, these coefficients are

$$K_{\mathrm{I}} + iK_{\mathrm{II}} = \sqrt{\pi c}\, p_0^+ [G_1(\theta_0, \varepsilon) + iG_2(\theta_0, \varepsilon)]$$

The values of $G_1(\theta_0, \varepsilon)$ and $G_2(\theta_0, \varepsilon)$ are presented in Table 1.

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ON THE DETERMINATION OF FORCES OF CONSTRAINT REACTION

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The equations of motion of mechanical systems with multipliers are reduced to the form enabling the separation of these equations into two groups, the first group describing the motions of the system, and the second group defining the multipliers. Each multiplier is determined independently of the remaining multipliers, and this makes it easy to assess the dynamic effect of each constraint on the system. On the basis of this approach, we study the following problems: determination of the constraint reactions [1], study of the motion of controlled systems with prescribed constraints [2, 3] and utilization of the method of nonholonomic mechanical systems in the case when the first integrals exist [4].

1. Equations of motion of a system with multipliers. We consider a system the position of which is defined in terms of the generalized coordinates q_i ($i = 1, 2, \ldots, n$). We assume that the system is restricted by ideal, nonholonomic, second order nonlinear constraints of the form

$$f_\alpha(t, q_i, q_i^{\cdot}, q_i^{\cdot\cdot}) = 0, \quad i = 1, 2, \ldots, n; \quad \alpha = 1, 2, \ldots, \Delta \tag{1.1}$$